On nontrivial zeros of the Riemann zeta function

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Abstract

The Riemann hypothesis on nontrivial zeros of the Riemann zeta function is proved. If a complex number $s_0 = \sigma_0 + it_0$ is a nontrivial zero, then (σ_0, t_0) is a solution to a system of two equations of two real variables σ and t. Considering one of that two equations one can found that one side of it increases and the other decreases as a function of $\sigma \in (0; 1)$ on the set of so called critical values of σ at the "height" $t = t_0$, so (σ_0, t_0) is the unique solution at $t = t_0$. As nontrivial zeros are symmetric about the line Re $s = 1/2$ it follows that $\sigma_0 = 1/2$.

Introduction and statement of the problem

Let $s = \sigma + it$ be a complex variable, where $\sigma = \operatorname{Re} s, t = \operatorname{Im} s$, and
$x \in \mathbb{R}$ be a real variable.

It is known [1] that for $\operatorname{Re} s > 0, s \neq 1$ the Riemann zeta function $\zeta(s)$ has the representation:

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int\limits_1^\infty \frac{\{x\}}{x^{s+1}} dx \tag{1}$$

Here $\{x\}$ denotes the fractional part of a number x.

Let us rewrite equality 1 as

$$\zeta(s) = s \left(\frac{1}{s-1} - \int\limits_1^\infty \frac{\{x\}}{x^{s+1}} dx \right)$$

Thus, to obtain nontrivial zeros of the function $\zeta(s)$ we must solve the following equation:

$$\int\limits_1^\infty \frac{\{x\}}{x^{s+1}} = \frac{1}{s-1} \tag{2}$$

We get two equations:

$$\frac{1}{x^{s+1}} = \frac{1}{x^{\sigma+1}} \left(\cos(t\ln x) - isin(t\ln x)\right),$$
$$\frac{1}{s-1} = \frac{\sigma-1}{(\sigma-1)^2+t^2} - i\frac{t}{(\sigma-1)^2+t^2}.$$

Therefore, equation 2 is equivalent to the following system:

$$\begin{cases} \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \cos(t\ln x)dx = \frac{\sigma-1}{(\sigma-1)^2+t^2}, \\ \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \sin(t\ln x)dx = \frac{t}{(\sigma-1)^2+t^2}. \end{cases} \tag{3}$$

It is known that nontrivial zeros are symmetric about the real axis, therefore it suffices to consider the case $t > 0$.

In the sequel, we always assume that $0 < \sigma < 1,\ t > 0$.

Definition 1. *The half-trip* $0 < \sigma < 1,\ t > 0$ *is called critical.*

In the sequel, an arbitrary nontrivial zero $s_0 = \sigma_0 + it_0$ is considered to be fixed.

The Riemann hypothesis states that $\sigma_0 = 1/2$.

Left and right sides of the equations of system 3

Let us introduce four useful functions as follows:

$$u_1(\sigma,t) = \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \cos(t\ln x)dx,$$
$$v_1(\sigma,t) = \int\limits_1^\infty \frac{\{x\}}{x^{\sigma+1}} \sin(t\ln x)dx,$$
$$u_2(\sigma,t) = \frac{\sigma-1}{(\sigma-1)^2+t^2},$$
$$v_2(\sigma,t) = \frac{t}{(\sigma-1)^2+t^2}.$$

Equality 2 can be represented as follows

$$u_1(\sigma,t) - iv_1(\sigma,t) = u_2(\sigma,t) - iv_2(\sigma,t).$$

Thus one can represent system 3 in the following way:

$$\begin{cases} u_1(\sigma,t) = u_2(\sigma,t), \\ v_1(\sigma,t) = v_2(\sigma,t). \end{cases} \tag{4}$$

If $s_0 = \sigma_0 + it_0$ is a nontrivial zero, then (σ_0, t_0) is a solution to system 4, therefore it is a solution to the second equation $v_1(\sigma,t) = v_2(\sigma,t)$ of the system. Further on, we study the behavior of both sides of this equation.

Lemma 1. *By holding $t_0 > 0$ constant, the function $w = v_2(\sigma, t_0)$ increases as a function of one variable σ.*

Proof. It follows from the inequality

$$\frac{\partial v_2}{\partial \sigma} = -\frac{2(\sigma-1)t}{(t^2+(\sigma-1)^2)^2} > 0$$

□

Lemma 2. *Let $v_1(\sigma, t_0) > 0$ and $v_1(\sigma', t_0) > 0$ for σ, σ' such that $0 < \sigma < \sigma' < 1$. This yields the inequality $v_1(\sigma, t_0) > v_1(\sigma', t_0)$.*

Proof. The function $\sin(t_0 \ln x)$ has zeros (its x-intercepts) $x_k = e^{\pi k/t_0}$, where $k \in \mathbb{Z}$. Moreover, as $x \geq 1$, $t_0 \ln x \geq 0$ and $t_0 \ln x = \pi k$ we get $k \in \mathbb{N} \cup \{0\}$. As the function e^x is monotone increasing and tending to infinity, for each $n = 1, 2, \ldots$ the interval $(n, n+1)$ of the x-axis contains finite set of zeros.[1]. Between any two consecutive zeros, the function will be either positive or negative. Therefore these zeros divide each interval into a finite set of sub-intervals where the function $\sin(t_0 \ln x)$ is positive or negative.

Denote U_n^+ the union of sub-intervals where the function is positive, and U_n^- - the union of sub-intervals where the function is negative in the interval $(n, n+1), n = 1, 2, \ldots$.

Denote $\Psi^+(\sigma, x)$ the function that coincides with the function $\{x\} \sin(t_0 \ln x)/x^{\sigma+1}$ on the set $\bigcup_{n=1}^{\infty} U_n^+$ and is equal to 0 on the set $\bigcup_{n=1}^{\infty} U_n^-$.

The function $\Psi^-(\sigma, x)$ is to be constructed in the same way.

The series that generates the function $v_1(\sigma, t_0)$ converges absolutely, therefore we get

$$v_1(\sigma, t_0) = \sum_{n=1}^{\infty} \int_n^{n+1} \Psi^+(\sigma, x)dx + \sum_{n=1}^{\infty} \int_n^{n+1} \Psi^-(\sigma, x)dx \tag{5}$$

[1] For example, computer calculation showed that for $t_0 = 100$ the interval (1; 2) contains 22 zeros.

Rewrite equation 5 as following

$$v_1(\sigma, t_0) = \int_1^{\infty} \Psi^+(\sigma, x)dx + \int_1^{\infty} \Psi^-(\sigma, x)dx. \tag{6}$$

By assumption $v_1(\sigma, t_0) > 0$, thus from equation 6 we get

$$0 < \int_1^{\infty} (-\Psi^-)(\sigma, x)dx < \int_1^{\infty} \Psi^+(\sigma, x)dx \tag{7}$$

Denote $\alpha = \sigma' - \sigma$. Reasoning the same way that led to equation 5, we get

$$v_1(\sigma', t_0) = \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^+(\sigma, x)dx + \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^-(\sigma, x)dx$$

Rewrite this equation as following

$$v_1(\sigma', t_0) = \int_1^{\infty} \frac{1}{x^\alpha}\Psi^+(\sigma, x)dx + \int_1^{\infty} \frac{1}{x^\alpha}\Psi^-(\sigma, x)dx.$$

Consider the difference $v_1(\sigma, t_0) - v_1(\sigma', t_0)$. Let us determine its sign.

$$\begin{aligned}
v_1(\sigma, t_0) - v_1(\sigma', t_0) = \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} \Psi^+(\sigma, x)dx + \sum_{n=1}^{\infty} \int_n^{n+1} \Psi^-(\sigma, x)dx - \\
&- \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^+(\sigma, x)dx - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^-(\sigma, x)dx = \\
&= \left(\sum_{n=1}^{\infty} \int_n^{n+1} \Psi^+(\sigma, x)dx - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^+(\sigma, x)dx \right) + \\
&+ \left(\sum_{n=1}^{\infty} \int_n^{n+1} \Psi^-(\sigma, x)dx - \sum_{n=1}^{\infty} \int_n^{n+1} \frac{1}{x^\alpha}\Psi^-(\sigma, x)dx \right) = \\
&= \sum_{n=1}^{\infty} \int_n^{n+1} \left(1 - \frac{1}{x^\alpha}\right) \Psi^+(\sigma, x)dx + \sum_{n=1}^{\infty} \int_n^{n+1} \left(1 - \frac{1}{x^\alpha}\right) \Psi^-(\sigma, x)dx = \\
&= \int_1^{\infty} \left(1 - \frac{1}{x^\alpha}\right) \Psi^+(\sigma, x)dx + \int_1^{\infty} \left(1 - \frac{1}{x^\alpha}\right) \Psi^-(\sigma, x)dx.
\end{aligned}$$

As $1 - 1/x^\alpha > 0$ so it follows from inequality 7 that

$$\int_1^{\infty} \left(1 - \frac{1}{x^\alpha}\right) (-\Psi^-)(\sigma, x)dx < \int_1^{\infty} \left(1 - \frac{1}{x^\alpha}\right) \Psi^+(\sigma, x)dx$$

Therefore

$$\int_{1}^{\infty}\left(1-\frac{1}{x^{\alpha}}\right)\Psi^{+}(\sigma,x)dx+\int_{1}^{\infty}\left(1-\frac{1}{x^{\alpha}}\right)\Psi^{-}(\sigma,x)dx>0$$

This means that $v_1(\sigma,t_0)-v_1(\sigma',t_0)>0$, hence $v_1(\sigma,t_0)>v_1(\sigma',t_0)$, Q.E.D. $\square$

It follows from Lemma 1 that all values of the function $w=v_2(\sigma,t_0)$, where $\sigma\in(0;1)$, belong to the interval $(t_0/(1+t_0^2),1/t_0)$.

In other words, the graph of the function $w=v_2(\sigma,t_0)$ lies entirely in the rectangle $0<\sigma<1,\ t_0/(1+t_0^2)<w<1/t_0$.

Further interest is only the part of the graph of the function $w=v_1(\sigma,t_0)$ that is contained in this rectangle.

Definition 2. *The rectangle* $0<\sigma<1,\ t_0/(1+t_0^2)<w<1/t_0$ *is called critical.*

Definition 3. *A value of* σ *is called critical if the point* $(\sigma,v_1(\sigma,t_0))$ *belongs to the critical rectangle.*

Thus the value σ_0 is critical.

The graphs of $w=v_1(\sigma,t_0)$ and $w=v_2(\sigma,t_0)$ intersect at the point $(\sigma_0,v_2(\sigma_0,t_0))$.

Lemma 3. *Holding the value* $t_0>0$ *constant, we get the function* $v_1(\sigma,t_0)$ *decreasing on the set of critical values of the variable* σ.

Proof. Let σ_1 and σ_2 be arbitrary critical values such that $\sigma_1<\sigma_2$. The function $v_1(\sigma,t_0)$ is positive at this values, hence it follows from Lemma 2 that $v_1(\sigma_1,t_0)>v_1(\sigma_2,t_0)$. The lemma is proved. $\square$

The proof of the Riemann hypothesis

Theorem. *Let* $s_0=\sigma_0+it_0$ *be a nontrivial zero of the Riemann zeta function; then* $\sigma_0=1/2$.

Proof. A nontrivial zero of the zeta function is a solution to equation 2, hence the pair (σ_0,t_0) satisfies system 4, and, in particular, its second equality.

From Lemmas 2 and 3 it follows that this pair is unique. Suppose $\sigma_0\neq 1/2$. It is known that nontrivial zeros are symmetric about the line Re $s=1/2$, hence there exists another zero $1-\sigma_0+it_0$ at the same "height" $t=t_0$, therefore the pair $(1-\sigma_0,t_0)$ satisfies the second equality as well.

This contradiction proves the theorem. $\square$

References

[1] Галочкин А.И.,Нестеренко Ю.В., Шидловский А.Б. Введение в теорию чисел, Изд-во Московского университета,1984