

A proof of the Riemann hypothesis on nontrivial zeros of the Riemann zeta function

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Abstract

The Riemann hypothesis on nontrivial zeros of the Riemann zeta function is proved.

A complex number $s_0 = \sigma_0 + it_0$ is a nontrivial zero iff (σ_0, t_0) is a solution to a system of two equations of two real variables σ and t .

Considering one of that two equations, we found that one side of it is strictly increasing and the other one is nonincreasing as functions on the set of so called *critical values* $\sigma \in (0; 1)$ at the "height" $t = t_0$, so (σ_0, t_0) is a unique solution at $t = t_0$. As nontrivial zeros are symmetric about the line $\operatorname{Re} s = \frac{1}{2}$, it follows that $\sigma_0 = \frac{1}{2}$.

Keywords: the Riemann hypothesis, zeta function, nontrivial zeros.

Setting the problem

Let $s = \sigma + it$ be a complex variable, where $\sigma = \operatorname{Re} s, t = \operatorname{Im} s$, and $x \in \mathbb{R}$ be a real variable.

For $\operatorname{Re} s > 0, s \neq 1$, it is known [1] that the Riemann zeta function $\zeta(s)$ can be expressed by the formula

$$\zeta(s) = 1 + \frac{1}{s-1} - s \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx. \quad (1)$$

Here, $\{x\}$ denotes the fractional part of a number x .

Let us rewrite equality 1 in the form

$$\zeta(s) = s \left(\frac{1}{s-1} - \int_1^{\infty} \frac{\{x\}}{x^{s+1}} dx \right).$$

Thus, to obtain nontrivial zeros of the function $\zeta(s)$, we must solve the following equation:

$$\int_1^{\infty} \frac{\{x\}}{x^{s+1}} = \frac{1}{s-1}. \quad (2)$$

This implies two equations:

$$\begin{aligned} \frac{1}{x^{s+1}} &= \frac{1}{x^{\sigma+1}} (\cos(t \ln x) - i \sin(t \ln x)), \\ \frac{1}{s-1} &= \frac{\sigma-1}{(\sigma-1)^2 + t^2} - i \frac{t}{(\sigma-1)^2 + t^2}. \end{aligned}$$

Therefore, equation 2 is equivalent to the following system:

$$\begin{cases} \int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \cos(t \ln x) dx = \frac{\sigma-1}{(\sigma-1)^2 + t^2}, \\ \int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \sin(t \ln x) dx = \frac{t}{(\sigma-1)^2 + t^2}. \end{cases} \quad (3)$$

It is known that nontrivial zeros are symmetric about the real axis, therefore we consider only the case $t > 0$.

We always assume that $0 < \sigma < 1$, $t > 0$.

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

The Riemann hypothesis states that $\sigma_0 = \frac{1}{2}$.

Left and right sides of the equations of system 3

Let us introduce four useful functions as follows:

$$\begin{aligned} u_1(\sigma, t) &= \int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \cos(t \ln x) dx, \\ v_1(\sigma, t) &= \int_1^{\infty} \frac{\{x\}}{x^{\sigma+1}} \sin(t \ln x) dx, \\ u_2(\sigma, t) &= \frac{\sigma-1}{(\sigma-1)^2 + t^2}, \\ v_2(\sigma, t) &= \frac{t}{(\sigma-1)^2 + t^2}. \end{aligned}$$

Equation 2 can be expressed as follows:

$$u_1(\sigma, t) - iv_1(\sigma, t) = u_2(\sigma, t) - iv_2(\sigma, t).$$

We represent system 3 in the form

$$\begin{cases} u_1(\sigma, t) = u_2(\sigma, t), \\ v_1(\sigma, t) = v_2(\sigma, t). \end{cases} \quad (4)$$

$s = \sigma + it$ is a nontrivial zero if and only if (σ, t) is a solution to system 4.

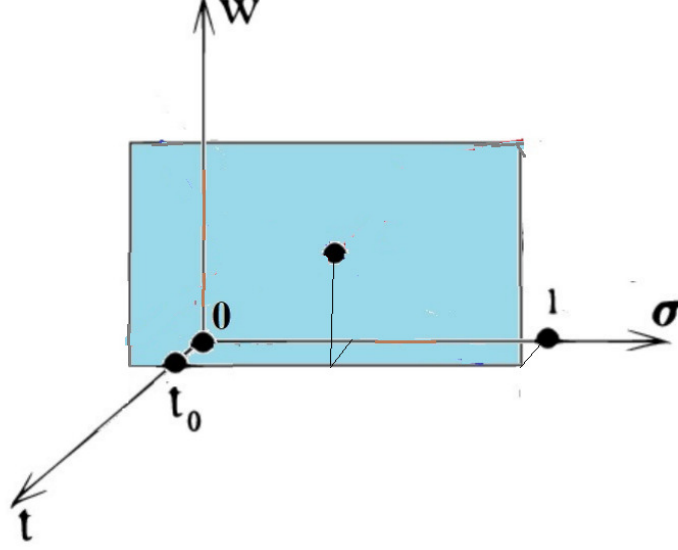


Figure 1: The plane $t = t_0$

Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero.

Lemma 1. *The function $w = v_2(\sigma, t_0)$ increases as a function of one variable $\sigma \in (0; 1)$.*

Proof. It follows from the inequality

$$\frac{dv_2}{d\sigma} = -\frac{2(\sigma - 1)t_0}{(t_0^2 + (\sigma - 1)^2)^2} > 0.$$

□

The range of the function $w = v_2(\sigma, t_0)$ is $U = \left(\frac{t_0}{1 + t_0^2}, \frac{1}{t_0} \right)$.

Obviously, the graph of the function $w = v_2(\sigma, t_0)$ lies in the rectangle $\Pi = \left\{ (\sigma, w) \mid \sigma \in (0; 1), w \in U \right\}$.

We consider the part of the graph of the function $v_1(\sigma, t_0)$ that lies in this rectangle.

Definition 1. *A rectangle Π is called critical.*

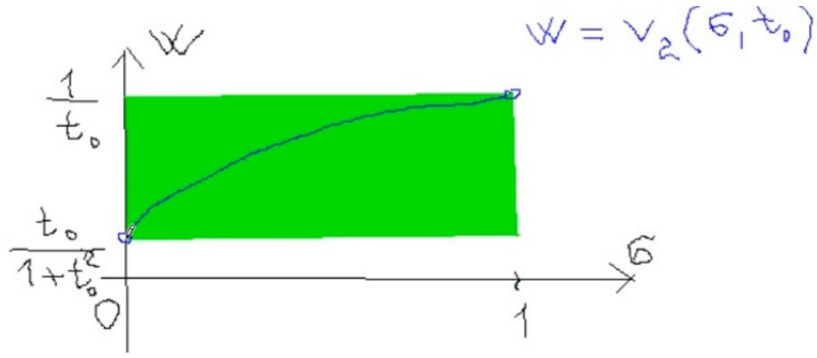


Figure 2: A critical rectangle

Remark 1. Critical rectangles are very thin, their width equals $\frac{1}{t_0} - \frac{t_0}{1+t_0^2} = \frac{1}{(1+t_0^2)t_0}$. Take the nontrivial zero with the least positive imaginary part $t_0 = 14.134725141\dots$ and get the width $0.0003523461812\dots$

Definition 2. σ is critical if $(\sigma, v_1(\sigma, t_0)) \in \Pi$.

Thus the value σ_0 is critical. The graphs of $v_1(\sigma, t_0)$ and $v_2(\sigma, t_0)$ intersect in the point $(\sigma_0, v_1(\sigma_0, t_0)) \in \Pi$.

This implies the inequality

$$v_1(\sigma_0, t_0) = \int_1^{+\infty} \frac{\{x\}}{x^{\sigma_0+1}} \sin(t_0 \ln x) dx = \frac{t_0}{\sigma_0^2 + t_0^2} > 0.$$

Moreover, by definition, we get $v_1(\sigma, t_0) \in \left(\frac{t_0}{1+t_0^2}, \frac{1}{t_0}\right)$ for all critical σ ; this implies that $v_1(\sigma, t_0) > 0$.

Let us introduce the function

$$\Psi(\sigma, x) = \frac{\{x\}}{x^{\sigma+1}} \sin(t_0 \ln x).$$

Then we have the equality

$$v_1(\sigma, t_0) = \int_1^{\infty} \Psi(\sigma, x) dx.$$

Lemma 2. The function $v_1(\sigma, t_0)$ does not increase on the set of all critical σ .

Proof. Let σ' be a positive number such that $\sigma + \sigma'$ is critical.

We must prove that $v_1(\sigma, t_0) \geq v_1(\sigma + \sigma', t_0)$.

It is obvious that

$$\Psi(\sigma + \sigma', x) = \frac{1}{x^{\sigma'}} \Psi(\sigma, x).$$

Then we get

$$v_1(\sigma + \sigma', t_0) = \int_1^\infty \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx.$$

Since σ and $\sigma + \sigma'$ are critical, we obtain $v_1(\sigma, t_0) > 0$ and $v_1(\sigma + \sigma', t_0) > 0$. This implies that there exists a X_0 such that for all $X > X_0$ we get the inequalities

$$\int_1^X \Psi(\sigma, x) dx > 0 \text{ and } \int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx > 0.$$

We must prove the inequality

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx \leq \int_1^X \Psi(\sigma, x) dx. \quad (5)$$

The proof consists of two parts.

Part 1

Let $\mathfrak{R}[a, b]$ be the set of Riemann-integrable functions on an interval $[a, b]$.

We use the following^[2]

Theorem (the second mean-value theorem for the integral¹). *If $f, g \in \mathfrak{R}[a, b]$ and g is a monotonic function on $[a, b]$, then there exists a point $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x)dx = g(a) \int_a^\xi f(x)dx + g(b) \int_\xi^b f(x)dx.$$

If $g(x) = \frac{1}{x^{\sigma'}}$ and $f(x) = \Psi(\sigma, x)$, then there exists a point $\xi = \xi(X) \in [1, X]$ such that

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x) dx = A + \gamma B,$$

$$\text{where } \gamma = \frac{1}{X^{\sigma'}}, A = A(\xi) = \int_1^\xi \Psi(\sigma, x) dx, \text{ and } B = B(\xi) = \int_\xi^X \Psi(\sigma, x) dx.$$

We have $0 < \gamma < 1, A + B > 0, A + \gamma B > 0$.

Let us prove inequality 5; this implies Lemma 2.

If $\xi = 1$, then $A = 0$. It follows from $A + \gamma B > 0$ that $\gamma B > 0$. As $B > 0$, we have $\gamma B < B$, and inequality 5 is true.

¹It states the equality which is often called Bonnet's formula

If $\xi = X$, then $B = 0$, we get $A + \gamma B = A$, and inequality 5 is true as well.

Assume that $1 < \xi < X$.

If $A \leq 0$, then $B > 0$, otherwise it would be $A + B \leq 0$. Inequality 5 is true as well.

If $A > 0$ and $B \geq 0$, then inequality 5 is true.

Case remained is $A > 0, B < 0$. In the sequel, it turns out to be impossible.

Part 2

Let us introduce the function $\Phi_1(x) = \int_x^\xi \Psi(\sigma, x)dx + \gamma B$, defined on $[1, \xi]$.

As $\Phi_1(1) > 0, \Phi_1(\xi) = B < 0$, there exists $\xi' \in (1, \xi)$ such that $\Phi_1(\xi') = 0$.

Then

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \underbrace{\int_1^{\xi'} \Psi(\sigma, x)dx + \int_{\xi'}^\xi \Psi(\sigma, x)dx + \gamma B}_A = \int_1^{\xi'} \Psi(\sigma, x)dx + \underbrace{\int_{\xi'}^\xi \Psi(\sigma, x)dx + \gamma B}_{\Phi_1(\xi')=0}.$$

We get

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx = \int_1^{\xi'} \Psi(\sigma, x)dx, \quad (6)$$

herewith

$$\int_{\xi'}^\xi \Psi(\sigma, x)dx + \gamma B = 0. \quad (7)$$

Now let us introduce the function $\Phi_2(x) = \int_x^X \Psi(\sigma, x)dx$.

As $B < 0$, we have $B < \gamma B$. It follows from this that

$$\Phi_2(\xi') = \int_{\xi'}^\xi \Psi(\sigma, x)dx + B < \int_{\xi'}^\xi \Psi(\sigma, x)dx + \gamma B = 0.$$

Simultaneously, $\Phi_2(\xi') < 0$ and $\Phi_2(1) > 0$, it follows from this that there exists a point $\xi'' \in (1, \xi')$ such that $\Phi_2(\xi'') = 0$.

So we get

$$\int_{\xi''}^X \Psi(\sigma, x)dx = 0. \quad (8)$$

Denote by $I(a, b)$ the integral $\int_a^b \Psi(\sigma, x)dx$.

$$\begin{aligned} \text{Then } 0 &= I(\xi'', X) = I(\xi'', \xi') + I(\xi', \xi) + I(\xi, X) = \\ &= I(\xi'', \xi') + \underbrace{I(\xi', \xi) + \gamma I(\xi, X)}_{=0} + (1 - \gamma)I(\xi, X), \end{aligned}$$

$$\text{we get } I(\xi'', \xi') + (1 - \gamma)I(\xi, X) = 0,$$

$$\text{it follows from this that } I(\xi'', X) = \underbrace{I(\xi'', \xi') + (1 - \gamma)I(\xi, X)}_{=0} + I(\xi', \xi) + \gamma I(\xi, X),$$

$$\text{thus } \int_{\xi''}^X \Psi(\sigma, x)dx = \int_{\xi'}^{\xi} \Psi(\sigma, x)dx + \gamma \int_{\xi}^X \Psi(\sigma, x)dx = 0.$$

$$\text{Consequently, } \int_1^{\xi''} \Psi(\sigma, x)dx = \int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx.$$

$$\text{Taking into account equality 6, we get } \int_1^{\xi'} \Psi(\sigma, x)dx = \int_1^{\xi''} \Psi(\sigma, x)dx, \text{ thus } \int_{\xi''}^{\xi'} \Psi(\sigma, x)dx = 0.$$

With equality 7 we get $\int_{\xi'}^{\xi} \Psi(\sigma, x)dx + B = \int_{\xi'}^{\xi} \Psi(\sigma, x)dx + \gamma B$, but then $B = \gamma B$, it follows from this that $B = 0$.

As $B < 0$, we got a contradiction, this implies that the case $A > 0, B < 0$ is impossible.

Thus, for arbitrary $X > X_0$ inequality 5 is true, consequently

$$\int_1^X \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx \leq \int_1^X \Psi(\sigma, x)dx.$$

We get the inequality

$$\int_1^{\infty} \frac{1}{x^{\sigma'}} \Psi(\sigma, x)dx \leq \int_1^{\infty} \Psi(\sigma, x)dx. \quad (9)$$

□

The proof of the Riemann hypothesis

Theorem. *Let $s_0 = \sigma_0 + it_0$ be a nontrivial zero of the Riemann zeta function; then $\sigma_0 = \frac{1}{2}$.*

Proof. A nontrivial zero of the zeta function is a solution to equation 2, hence the pair (σ_0, t_0) satisfies system 4, and, in particular, its second equality.

From Lemma 2 it follows that this pair is unique. Suppose $\sigma_0 \neq \frac{1}{2}$.

It is known that nontrivial zeros are symmetric about the line $\text{Re } s = \frac{1}{2}$, hence there exists

another zero $1 - \sigma_0 + it_0$ at the same "height" $t = t_0$, therefore the pair $(1 - \sigma_0, t_0)$ satisfies the second equality as well.

This contradiction establishes the theorem. □

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